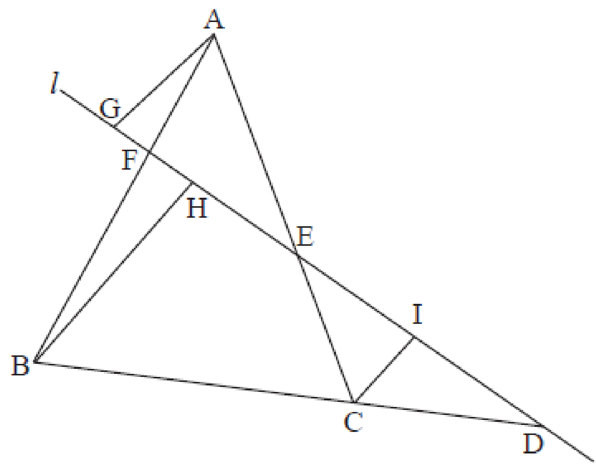
SL Paper 2



a. The diagram shows the line l meeting the sides of the triangle ABC at the points D, E and F. The perpendiculars to l from A, B and C meet l at G, H and I. [13]

(i) State why $\frac{\text{AF}}{\text{FB}} = \frac{\text{AG}}{\text{HB}}$.

(ii) Hence prove Menelaus' theorem for the triangle ABC.

(iii) State and prove the converse of Menelaus' theorem.

b. A straight line meets the sides (PQ), (QR), (RS), (SP) of a quadrilateral PQRS at the points U, V, W, X respectively. Use Menelaus' theorem to show that [7]

$$\frac{\text{PU}}{\text{UQ}} \times \frac{\text{QV}}{\text{VR}} \times \frac{\text{RW}}{\text{WS}} \times \frac{\text{SX}}{\text{XP}} = 1.$$

The circle C has centre O. The point Q is fixed in the plane of the circle and outside the circle. The point P is constrained to move on the circle.

A.a Show that the opposite angles of a cyclic quadrilateral add up to 180° . [3]

A.b A quadrilateral ABCD is inscribed in a circle S . The four tangents to S at the vertices A, B, C and D form the edges of a quadrilateral EFGH. Given that EFGH is cyclic, show that AC and BD intersect at right angles. [7]

B.a Show that the locus of a point P′ , which satisfies $\overrightarrow{\text{QP}'} = k\overrightarrow{\text{QP}}$, is a circle C' , where $\boldsymbol{k}$ is a constant and $0 < k < 1$. [6]

B.b Show that the two tangents to C from Q are also tangents to C′ . [4]

A circle C passes through the point $(1,\ 2)$ and has the line $3x - y = 5$ as the tangent at the point $(3,\ 4)$.

a. Find the coordinates of the centre of C and its radius. [9]

b. Write down the equation of C. [1]

c. Find the coordinates of the second point on C on the chord through $(1,\ 2)$ parallel to the tangent at $(3,\ 4)$. [5]

The area of an equilateral triangle is 1 cm^2. Determine the area of:

The points A, B have coordinates (1, 0), (0, 1) respectively. The point P(x, y) moves in such a way that $\text{AP} = k\text{BP}$ where $k \in \mathbb{R}^+$.

A.a. the circumscribed circle. [8]

A.b. the inscribed circle. [3]

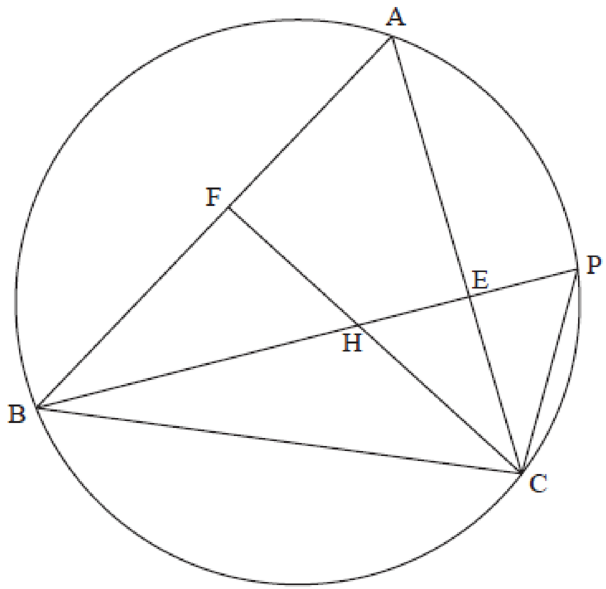
B.a. When $k = 1$, show that the locus of P is a straight line. [3]

B.b. When $k \neq 1$, the locus of P is a circle. [9]

(i) Find, in terms of k , the coordinates of C, the centre of this circle.

(ii) Find the equation of the locus of C as k varies.

In the acute angled triangle ABC, the points E, F lie on [AC], [AB] respectively such that [BE] is perpendicular to [AC] and [CF] is perpendicular to [AB]. The lines (BE) and (CF) meet at H. The line (BE) meets the circumcircle of the triangle ABC at P. This is shown in the following diagram.



a. (i) Show that CEFB is a cyclic quadrilateral. [7]

(ii) Show that $\mathrm{HE} = \mathrm{EP}$.

b. The line (AH) meets [BC] at D. [8]

(i) By considering cyclic quadrilaterals show that $\mathrm{C\hat{A}D} = \mathrm{E\hat{F}H} = \mathrm{E\hat{B}C}$.

(ii) Hence show that [AD] is perpendicular to [BC].

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a. Given that the elements of a 2×2 symmetric matrix are real, show that [11]

(i) the eigenvalues are real;

(ii) the eigenvectors are orthogonal if the eigenvalues are distinct.

b. The matrix $\boldsymbol{A}$ is given by [7]

$$\boldsymbol{A} = \begin{pmatrix} 11 & \sqrt{3} \\ \sqrt{3} & 9 \end{pmatrix}.$$

Find the eigenvalues and eigenvectors of $\boldsymbol{A}$.

c. The ellipse E has equation $\boldsymbol{X}^T\boldsymbol{AX} = 24$ where $\boldsymbol{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\boldsymbol{A}$ is as defined in part (b). [7]

(i) Show that E can be rotated about the origin onto the ellipse E' having equation $2x^2 + 3y^2 = 6$.

(ii) Find the acute angle through which E has to be rotated to coincide with E' .

Figure 1

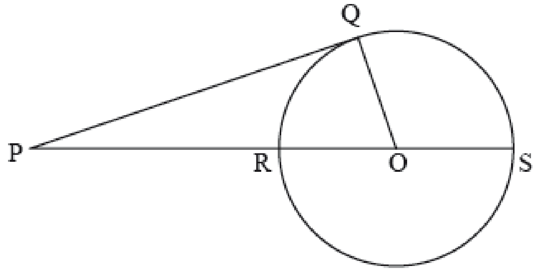


Figure 1 shows a tangent [PQ] at the point Q of a circle and a line [PS] meeting the circle at the points R , S and passing through the centre O of the circle.

Figure 2

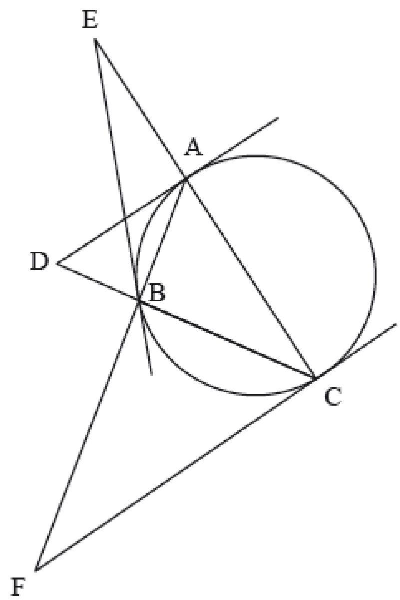


Figure 2 shows a triangle ABC inscribed in a circle. The tangents at the points A , B , C meet the opposite sides of the triangle externally at the points D , E , F respectively.

a.i. Show that $\mathrm{PQ}^2 = \mathrm{PR} \times \mathrm{PS}.$ [2]

a.ii. State briefly how this result can be generalized to give the tangent-secant theorem. [2]

b.i. Show that $\frac{\mathrm{AD}^2}{\mathrm{BD}^2} = \frac{\mathrm{CD}}{\mathrm{BD}}.$ [2]

b.ii. By considering a pair of similar triangles, show that [2]

$$\frac{\mathrm{AD}}{\mathrm{BD}} = \frac{\mathrm{AC}}{\mathrm{AB}} \text{ and hence that } \frac{\mathrm{CD}}{\mathrm{BD}} = \frac{\mathrm{AC}^2}{\mathrm{AB}^2}.$$

b.iii. By writing down and using two further similar expressions, show that the points D, E, F are collinear. [6]

Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

The area enclosed by the ellipse is 8π and $b = 2.$

a. Show that the area enclosed by the ellipse is πab. [9]

b.i. Determine which coordinate axis the major axis of the ellipse lies along. [2]

b.ii. Hence find the eccentricity. [2]

b.iii. Find the coordinates of the foci. [2]

b.iv. Find the equations of the directrices. [2]

c. The centre of another ellipse is now given as the point (2, 1). The minor and major axes are of lengths 3 and 5 and are parallel to the x and y axes respectively. Find the equation of the ellipse. [3]

Consider the ellipse having equation $x^2 + 3y^2 = 2$.

a. (i) Find the equation of the tangent to the ellipse at the point $\left(1,\ \frac{1}{\sqrt{3}}\right)$. [7]

(ii) Find the equation of the normal to the ellipse at the point $\left(1,\ \frac{1}{\sqrt{3}}\right)$.

b. Given that the tangent crosses the x-axis at P and the normal crosses the y-axis at Q, find the equation of (PQ). [4]

c. Hence show that (PQ) touches the ellipse. [4]

d. State the coordinates of the point where (PQ) touches the ellipse. [1]

e. Find the coordinates of the foci of the ellipse. [4]

f. Find the equations of the directrices of the ellipse. [1]

The hyperbola with equation $x^2 - 4xy - 2y^2 = 3$ is rotated through an acute anticlockwise angle α about the origin.

a. The point $(x,\ y)$ is rotated through an anticlockwise angle α about the origin to become the point $(X,\ Y)$. Assume that the rotation can be represented by [3]

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Show, by considering the images of the points $(1,\ 0)$ and $(0,\ 1)$ under this rotation that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

b.i. By expressing $(x,\ y)$ in terms of $(X,\ Y)$, determine the equation of the rotated hyperbola in terms of X and Y. [3]

b.ii. Verify that the coefficient of XY in the equation is zero when $\tan\alpha = \frac{1}{2}$. [3]

b.iii Determine the equation of the rotated hyperbola in this case, giving your answer in the form $\frac{X^2}{A^2} - \frac{Y^2}{B^2} = 1$. [3]

b.iv Hence find the coordinates of the foci of the hyperbola prior to rotation. [5]

The points D, E, F lie on the sides [BC], [CA], [AB] of the triangle ABC and [AD], [BE], [CF] intersect at the point G. You are given that $\text{CD} = 2\text{BD}$ and $\text{AG} = 2\text{GD}$.

A.a By considering (BE) as a transversal to the triangle ACD, show that [2]

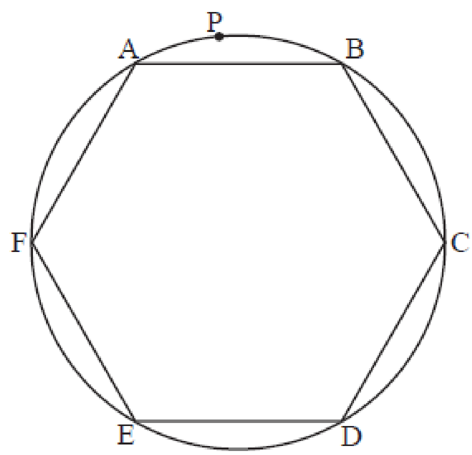
$$\frac{\text{CE}}{\text{EA}} = \frac{3}{2}.$$

A.b Determine the ratios [7]

(i) $\frac{\text{AF}}{\text{FB}}$;

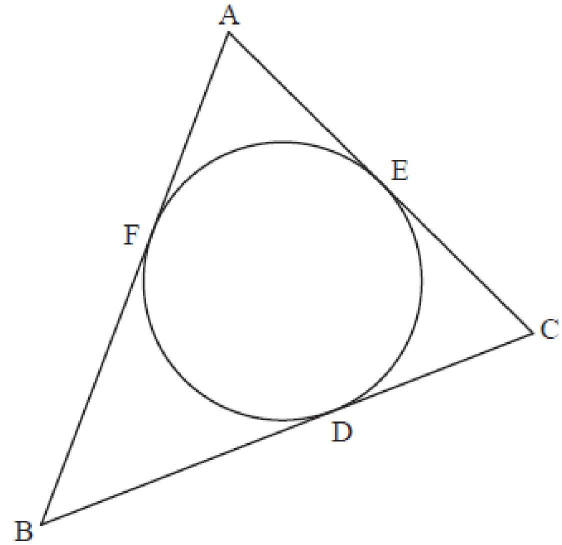
(ii) $\frac{\text{BG}}{\text{GE}}$.

B. [7]



The diagram shows a hexagon ABCDEF inscribed in a circle. All the sides of the hexagon are equal in length. The point P lies on the minor arc AB of the circle. Using Ptolemy’s theorem, show that

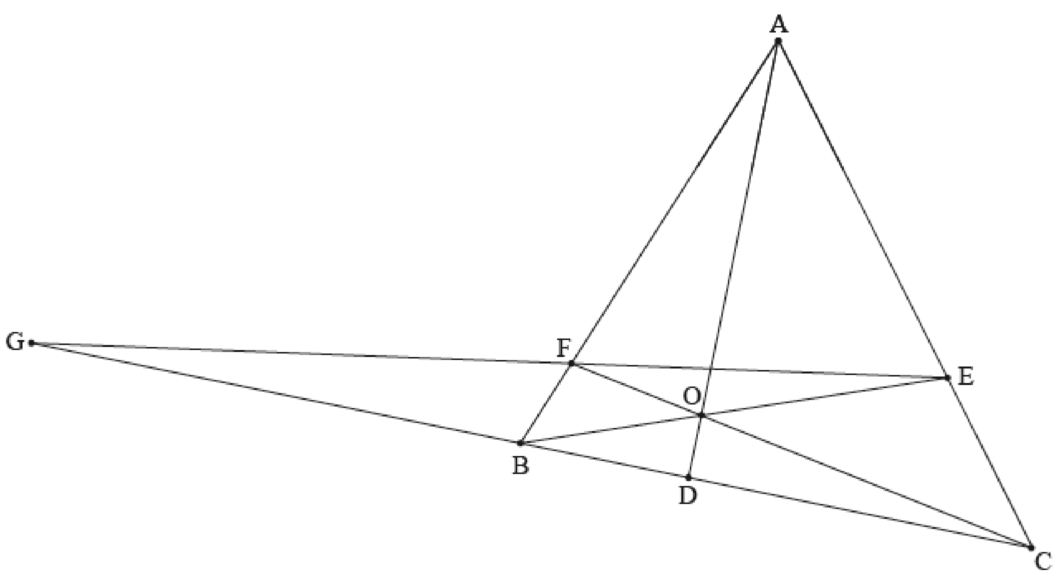
$$\text{PE} + \text{PD} = \text{PA} + \text{PB} + \text{PC} + \text{PF}$$



a. The diagram shows triangle ABC together with its inscribed circle. Show that [AD], [BE] and [CF] are concurrent. [8]

b. PQRS is a parallelogram and T is a point inside the parallelogram such that the sum of $\text{P}\hat{\text{T}}\text{Q}$ and $\text{R}\hat{\text{T}}\text{S}$ is 180°. Show that [13]

$\text{TP} \times \text{TR} + \text{ST} \times \text{TQ} = \text{PQ} \times \text{QR}$.



The diagram above shows a point O inside a triangle ABC. The lines (AO), (BO), (CO) meet the lines (BC), (CA), (AB) at the points D, E, F respectively. The lines (EF), (BC) meet at the point G.

(a) Show that, with the usual convention for the signs of lengths in a triangle, $\frac{\text{BD}}{\text{DC}} = -\frac{\text{BG}}{\text{GC}}$.

(b) The lines (FD), (CA) meet at the point H and the lines (DE), (AB) meet at the point I. Show that the points G, H, I are collinear.

A. Prove that the interior bisectors of two of the angles of a non-isosceles triangle and the exterior bisector of the third angle, meet the sides of the triangle in three collinear points. [8]

B.a An equilateral triangle QRT is inscribed in a circle. If S is any point on the arc QR of the circle, [10]

(i) prove that $\mathrm{ST} = \mathrm{SQ} + \mathrm{SR}$;

(ii) show that triangle RST is similar to triangle PSQ where P is the intersection of [TS] and [QR];

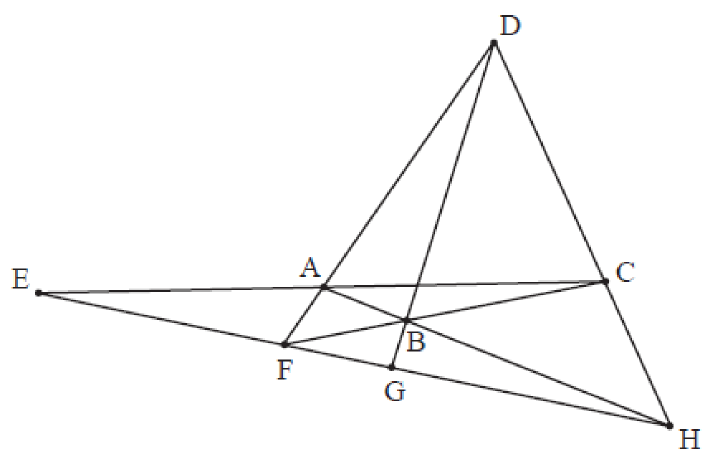
(iii) using your results from parts (i) and (ii) deduce that $\frac{1}{\mathrm{SP}} = \frac{1}{\mathrm{SQ}} + \frac{1}{\mathrm{SR}}$.

B.b Perpendiculars are drawn from a point P on the circumcircle of triangle LMN to the three sides. The perpendiculars meet the sides [LM], [MN] and [LN] at the points E, F and G respectively. [8]

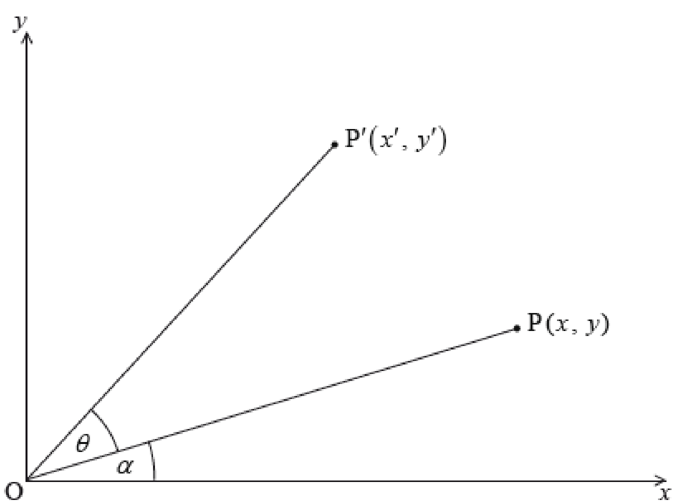
Prove that $\mathrm{PL} \times \mathrm{PF} = \mathrm{PM} \times \mathrm{PG}$.

ABCD is a quadrilateral. (AD) and (BC) intersect at F and (AB) and (CD) intersect at H. (DB) and (CA) intersect (FH) at G and E respectively.

This is shown in the diagram below.



Prove that $\frac{\mathrm{HG}}{\mathrm{GF}} = -\frac{\mathrm{HE}}{\mathrm{EF}}$.



The diagram above shows the points $\mathrm{P}(x,\ y)$ and $\mathrm{P}'(x',\ y')$ which are equidistant from the origin O. The line (OP) is inclined at an angle α to the x-axis and $\mathrm{P\hat{O}P'} = \theta$.

(a) (i) By first noting that $\text{OP} = x \sec \alpha$, show that $x' = x \cos \theta - y \sin \theta$ and find a similar expression for y'.

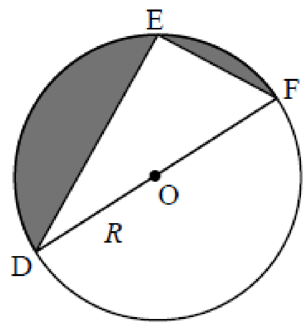
(ii) Hence write down the 2×2 matrix which represents the anticlockwise rotation about O which takes P to P'.

(b) The ellipse E has equation $5x^2 + 5y^2 - 6xy = 8$.

(i) Show that if E is rotated **clockwise** about the origin through 45°, its equation becomes $\frac{x^2}{4} + y^2 = 1$.

(ii) Hence determine the coordinates of the foci of E.

A new triangle DEF is positioned within a circle radius *R* such that DF is a diameter as shown in the following diagram.



a.i. In a triangle ABC, prove $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$. [4]

a.ii. Prove that the area of the triangle ABC is $\frac{1}{2} ab \sin \text{C}$. [2]

a.iii. Given that *R* denotes the radius of the circumscribed circle prove that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$. [2]

a.iv. Hence show that the area of the triangle ABC is $\frac{abc}{4R}$. [2]

b.i. Find in terms of *R*, the two values of $(\text{DE})^2$ such that the area of the shaded region is twice the area of the triangle DEF. [9]

b.ii. Using two diagrams, explain why there are two values of $(\text{DE})^2$. [2]

c. A parallelogram is positioned inside a circle such that all four vertices lie on the circle. Prove that it is a rectangle. [3]